Equivalent Transmission Lines for Quantum Particles in Sectionally Constant Potentials

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Introduction

- The motion of a nonrelativistic spinless quantum particle moving on a straight line under a sequence of constant potentials is modeled by a cascade of generalized transmission lines.
- The equivalence between physical quantities of the system and electrical quantities of the model is illustrated.
- The equivalent circuit can be used for simulation and as an help to understand quantum properties.

The physical description

The physical description of the system is embedded in the Schrödinger equation

$$\mathrm{i}\hbar\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\psi(x,t) + V(x)\psi(x,t) \qquad (1)$$

where the potential energy $V(x)$ is assumed to be piecewise constant in x and independent on t.
$\psi(x, T)$ is the *wavefunction*, the squared modulus of which defines the probability that the particle is found at point x, if its position is measured at time t.

The physical description

The solutions of eq.(1) have the form

$$\psi(x,t) = C(k)\mathrm{e}^{\mathrm{i}(\gamma x - zt)} \tag{2}$$

where γ and z are the *complex wavenumber* and *complex radian frequency* defined as

$$\gamma = k + \mathrm{i}\alpha, \quad z = \omega + \mathrm{i}\tau \tag{3}$$

We assume that the particle is in a state of well defined energy, that, due to the well known relation $E = \hbar\omega$, means states of well defined real radian frequency; thus we will use $z = \omega$.

The physical description

By introducing the solution of equation (2) in equation (1) with the above proviso, the *dispersion equation* is obtained

$$\hbar\omega = \frac{(\hbar\gamma)^2}{2m} + V \tag{4}$$

which gives for γ the solution

$$\gamma = \sqrt{\frac{2m}{\hbar}(\omega - \omega_0)} = \begin{cases} k & \text{for} \quad \omega > \omega_0 \\ \mathrm{i}\alpha & \text{for} \quad \omega < \omega_0 \end{cases} \qquad \omega_0 = \frac{V}{\hbar} \tag{5}$$

The solution, harmonic in time, can be either harmonic or exponential in space.

Transmission line models

Each piece of eq. (1) in which V is constant will be represented by a piece of uniform lossless transmission line of the same length. The whole system is represented by the ordered cascade of such sections.

Since conventional transmission lines include only real elements, the complex form of eq.(1) demands for the introduction of a lossless complex element, the *imaginary resistor* [3], i.e. a frequency independent reactor of impedance $z = \mathrm{i}x$ with x frequency independent.

Transmission line models

The transmission line equations at a fixed radian frequency ω are

$$-\frac{\mathrm{d}U(x,\omega)}{\mathrm{d}x} = z'(\omega)I(x,\omega)$$
$$-\frac{\mathrm{d}I(x,\omega)}{\mathrm{d}x} = y'(\omega)V(x.\omega) \tag{6}$$

The complex *wave number* and emphcharacteristic impedance are in turn

$$\gamma = \mathrm{i}\sqrt{z'y'}, \qquad Z_\mathrm{c} = \sqrt{\frac{z'}{y'}} \tag{7}$$

The solutions of the equations above are written

$$U(x,\omega) = \left(A\mathrm{e}^{\mathrm{i}\gamma x} + B\mathrm{e}^{-\mathrm{i}\gamma x}\right)\mathrm{e}^{-\mathrm{i}\omega t}$$
$$I(x,\omega) = \sqrt{\frac{1}{Z_\mathrm{c}}}\left(A\mathrm{e}^{\mathrm{i}\gamma x} - B\mathrm{e}^{-\mathrm{i}\gamma x}\right)\mathrm{e}^{-\mathrm{i}\omega t} \tag{8}$$

Transmission line models

The quantities z' and y' must be chosen in such a way as to satisfy the first of eqs.(7) under the condition of eq.(5).

It is evident that one of them can be chosen arbitrarily, while the other is uniquely defined.

The arbitrary choice should anyway to obtain a solution of minimum degree to obtain the simplest equivalent circuit.

Transmission line models

Choose

$$z' = -\mathrm{i} \quad U \doteq \psi \quad \text{and} \quad I = -\frac{1}{z}\frac{\partial U}{\partial x} \doteq -\mathrm{i}\frac{\partial \psi}{\partial x} \tag{9}$$

From equations (5), (7), and (9) we obtain

$$y' = -\frac{k^2}{z'} \doteq -\mathrm{i}\frac{2m}{\hbar}(\omega - \omega_0) \tag{10}$$

The discretized structure describing the transmission line is shown in Figure 1.

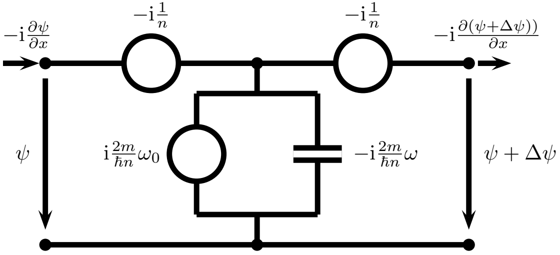

Figure: Discrete cell of pseudo–TE structure.

Transmission line models

Choose

$$y' = -\mathrm{i} \quad \text{and} \quad I \doteq \psi, \quad V = -\frac{1}{y}\frac{\partial I}{\partial x} \doteq -\mathrm{i}\frac{\partial \psi}{\partial x} \tag{11}$$

From equations (5), (7), and (9) we obtain

$$z' = -\frac{k^2}{y'} \doteq -\mathrm{i}\frac{2m}{\hbar}(\omega - \omega_0) \tag{12}$$

The discretized structure describing the transmission line is shown in Figure 2.

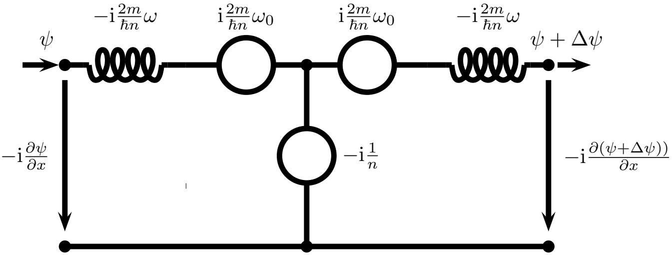

Figure: Discrete cell of pseudo–TM structure.

Power and probability

Each section i of the chain is *transparent* or *opaque* according to $\omega \gtrless \omega_{0i}$.

For transparent sections, from eq.(8) the active and reactive powers are calculated as

$$\begin{aligned} P &= k(|A!^2 - |B|^2) \\ Q &= -\mathrm{i}k(AB^*\mathrm{e}^{\mathrm{i}2kx} - BA^*\mathrm{e}^{-\mathrm{i}2kx}) \end{aligned} \tag{13}$$

For the opaque

$$\begin{aligned} P &= \mathrm{i}\alpha(AB^* - BA^*) \\ Q &= -\alpha(|A|^2\mathrm{e}^{-2\alpha x} - |B|^2\mathrm{e}^{2\alpha x}) \end{aligned} \tag{14}$$

Power and probability

Density probability that the quantum particle traverses a transparent section

$$J = -\frac{\mathrm{i}\hbar}{2m}\left(\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi *}{\partial x}\right) = \frac{\hbar k}{m}(|A|^2 - |B|^2)) \qquad (15)$$

or

$$P \doteq \frac{m}{\hbar} J \qquad (16)$$

Density of probability that the particle is found at x: $\rho(x) = \psi^*(x)\psi(x)$

$$Q \doteq -\frac{1}{2}\frac{\partial \rho}{\partial x} \qquad (17)$$

Since ρ does not depend on t and J does not depend on x, the continuity equation is identically verified.

Power and probability

The case of an opaque barrier gives the results

$$J = \frac{\mathrm{i}\hbar\alpha}{m}(AB^* - BA^*) \tag{18}$$

and

$$\rho = |A|^2 \mathrm{e}^{-2\alpha x} + |B|^2 \mathrm{e}^{2\alpha x} + (BA^* - AB^*) \tag{19}$$

so that equations (16) and (17) still apply.

Conclusion

We have shown that the dynamics of a quantum particle traversing a cascade of constante potentials of arbitrary amplitude and width can be represented by a cascade of transmission lines including the classical lossless elements plus the frequency independent reactance (or imaginary resistor).

References

P. P. Civalleri, M. Gilli and M. Bonnin, "A Transmission Line Model for the Free Electron–Positron Field", *IEEE Nano 2010*, August 17–20, 2010, Kintex (Korea), ICO-TS07-027.

Merzbacher, Eugen, *Quantum Mechanics*, 3rd ed., Wiley, Noboken (NJ, USA), 1998, p. 92.

Belevitch, Vitold, *Classical Network Theory*, S. Francisco (CA, USA), Holden–Day, 1968, p. 57.